was found. Taking the second of the resonance relations $(\nu = \beta = 2)$, as the fundamental resonance, we obtain $|P_1| = 3 > 1$. But this means that the case of interaction considered here satisfies all the conditions of Theorem 3; therefore the libration points shown are unstable.

REFERENCES

1. KUNITSYN A.L. and MARKEYEV A.P., Stability in resonant cases. In: Achievements of Science and Technology. General Mechanics. Vol.4, Moscow, VINITI, 1979.
2. KHAZIN L.G. and SHNOL' E.E., Equilibrium stability conditions under 1:3 resonance. PMM, Vol.44, No.2, 1980.
3. KUNITSYN A.L., On the asymptotic stability of resonant systems. PMM Vol.42, No.6, 1978.
4. KUNITSYN A.L., MEDVEDEV S.V. and TKHAI V.N., Derivation of an algorithm and writing a program for solving certain resonance problems of the stability of dynamic systems on a digital computer. In: Problems of Stability of Motion, Analytical Mechanics and Motion Control. Novosibirsk, Nauka, 1979.
5. MOLCHANOV A.M., Stability in the case of neutrality of the linear approximation. Dokl. Akad. Nauk SSSR, Vol.141, No.1, 1961.
6. VERETENNIKOV V.G., On the stability of motion in the case of three pairs of purely imaginary roots. Tr. Un-ta Druzhby narodov im. Patrisa Lumumby, Ser. teor. mekhan., Vol.15, No.3, 1966.
7. RADZIEVSKII V.V., Bounded three-body problem taking light pressure into acount. Astron. Zh. Vol.27, No.4, 1950.
8. RADZIEVSKII V.V., Three-dimensional case of a bounded problem of three radiating and gravitating bodies. Astron. Zh. Vol.30, No.3, 1953.
9. PEREZHOGIN A.A., On the stability of libration points in a bounded, photogravitational circular three-body problem. Kosmich. Issledovaniya, Vol.20, No.2, 1982.

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SMALL VIBRATIONS OF ONE-DIMENSIONAL MOVING BODIES*

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Problems of the transverse vibrations of moving strings, hoses with flowing liquid, as well as bodies that can be represented in the form of a set of interacting strings moving at different velocities are examined. It is assumed that there are not tangential stresses between the strings. The vibrations are described by a second-order linear differential equation whose coefficients are obtained by summing the corresponding parameters of the separate strings. The distinctive feature of this kind of system is the difference in the wave propagation velocities in the forward and reverse directions.

A transformation is presented that enables the problem of vibrational processes in a moving body with conditions given on fixed boundaries to be reduced to a boundary value problem for a string at rest. Questions concerning the critical velocities, the free vibration energy of the moving body, and the type of dissipative term are considered. Analytic solutions are given for problems regarding free vibrations and the steady-state regime of forced vibrations under the action of a force varying sinusoidally with time.

1. Formulation of the problem. In a linear approximation we will consider the transverse vibrations of a body (or system of bodies) moving uniformly and rectilinearly along the x axis in the ground state. In the simplest case, the equation of a taut filament (string) moving at a velocity v is obtained from the equations of the string at rest

$$\rho u_{tt} - T u_{xx} = F \tag{1.1}$$

(the notation is standard) by replacing the partial derivative with respect to the time $\partial/\partial t$ by the substantive derivative $\partial/\partial t + v\partial/\partial x$

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$$\rho u_{tt} + 2\rho v u_{tx} + (\rho v^2 - T)\, u_{xx} = F \tag{1.2}$$

or

$$u_{tt} + 2v u_{tx} + (v^2 - c_0^2)\, u_{xx} = f, \quad c_0^2 = T/\rho, \ f = F/\rho \tag{1.3}$$

Problems of the vibrations of bodies comprised of two filaments, strings with a moving distributed inertial load, and a hose, a stretched flexible pipe with a flowing ideal fluid /1, 2/, are examined on the same mathematical basis. The fluid is here considered to be either an inertial laod, or the pressure drop is also taken into account, which is equivalent to the appearance of negative tension. It is assumed that there are no tangential stresses between the strings.

The equations for the combined vibrations of such a system can be obtained as follows. We denote the force acting on the first string by the second by F_{12}, we similarly introduce F_{21}, we also furnish the variables in (1.2) with the subscripts 1 and 2. Taking $u_1 = u_2 = u$, and using the equation $F_{12} = -F_{21}$, we obtain

$$(\rho_1 + \rho_2)\, u_{tt} + 2\,(\rho_1 v_1 + \rho_2 v_2)\, u_{tx} + (\rho_1 v_1^2 + \rho_2 v_2^2 - T_1 - T_2)\, u_{xx} = 0 \tag{1.4}$$

If the wall of the hose is considered to be fixed, $v_2 = 0$ and the pressure drop (in a finite interval) is denoted by Π, we obtain the equation normally used in practice

$$(\rho_1 + \rho_2)\, u_{tt} + 2\rho_1 v_1 u_{tx} + (\rho_1 v_1^2 - T_1 + \Pi)\, u_{xx} = 0 \tag{1.5}$$

We have similarly for n strings under the conditions $F_{jk} = -F_{kj}$:

$$Ru_{tt} + 2Pu_{tx} + (K - \theta)\, u_{xx} = 0 \tag{1.6}$$
$$R = \sum_{k=1}^{n} \rho_k, \quad P = \sum_{k=1}^{n} \rho_k v_k, \quad K = \sum_{k=1}^{n} \rho_k v_k^2, \quad \theta = \sum_{k=1}^{n} T_k$$

An infinite set of strings can formally be considered, and the sum can be replaced by an integral over a segment or a two-dimensional domain

$$R = \int \rho\,(\alpha)\, d\alpha, \quad P = \int \rho\,(\alpha)\, v\,(\alpha)\, d\alpha, \quad K = \int \rho\,(\alpha)\, v^2\,(\alpha)\, d\alpha$$
$$\theta = \int T\,(\alpha)\, d\alpha$$

In particular, taking into account the dispersion in the velocities in a flowing fluid we obtain an equation for the hose ($\langle \ldots \rangle$ is the mean value)

$$(\rho_1 + \rho_2)\, u_{tt} + 2\rho_1 \langle v \rangle\, u_{tx} + (\rho_1 \langle v^2 \rangle - T_1)\, u_{xx} = 0$$

In fact, we can speak of an approximate description of the vibrations of a two- or three-dimensional body in these models, whose transverse dimensions are small compared with the longitudinal dimensions as well as with the wavelength. The body is represented as a set of strings, where the parameters of the separate strings enter the appropriate body parameters - the coefficients of (1.6) - additively.

Consider the following example. Suppose we have a layer $-h \leqslant y \leqslant h$ of an ideal incompressible fluid of density ρ flowing with velocity v in the direction of the x axis. The layer is bounded by two membranes. For $y = h$ the membrane is of density ρ_1 with tension T_1 moving at a velocity v_1 while the parameters of the second membrane for $y = -h$ will be ρ_2, T_2, v_2, respectively. . We will examine the problem of small system vibrations in the xy plane. In this case it is also possible to speak about the vibrations of two strings interacting with an infinite set of liquid filaments. If there are no external forces, the fluid flow will be potential and the problem can be solved by methods described in /3/. The potential φ will satisfy the equation $\Delta\varphi = 0$.

Denoting the membrane deflections by u_1 and u_2, we have the kinematic conditions on the lines $y = \pm h$ (there is slip on the fluid and membrane boundaries)

$$\frac{\partial u_1}{\partial t} + v\frac{\partial u_1}{\partial x} = \frac{\partial\varphi}{\partial y}\Big|_{y=h}, \quad \frac{\partial u_2}{\partial t} + v\frac{\partial u_2}{\partial x} = \frac{\partial\varphi}{\partial y}\Big|_{y=-h} \tag{1.7}$$

The pressure in the fluid can be expressed in terms of the potential by the formula

$$p = -\rho\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x}\right)$$

The dynamic conditions on the boundaries, the balance of pressure and forces-acting from the membrane side, have the form

$$-\rho\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x}\right)\Big|_{y=h} + T_1\frac{\partial^2 u_1}{\partial x^2} - \rho_1\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x}\right)^2 u_1 = 0 \tag{1.8}$$
$$\rho\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x}\right)\Big|_{y=-h} + T_2\frac{\partial^2 u_2}{\partial x^2} - \rho_2\left(\frac{\partial}{\partial t} + v_2\frac{\partial}{\partial x}\right)^2 u_2 = 0$$

We shall seek the solution of the problem in the form

$$u_n = b_n e^{i(\omega t - \varkappa x)},\ n = 1, 2;\ \varphi = (A \operatorname{sh} \varkappa y + B \operatorname{ch} \varkappa y) e^{i(\omega t - \varkappa x)} \tag{1.9}$$

where ω is the frequency, and $\varkappa$ is the wave number. We then obtain equations for the amplitudies from (1.7) and (1.8)

$$\begin{aligned} &ib_1 (\omega + \varkappa v) = \varkappa (A \operatorname{ch} \varkappa h + B \operatorname{sh} \varkappa h) \\ &ib_2 (\omega + \varkappa v) = \varkappa (A \operatorname{ch} \varkappa h - B \operatorname{sh} \varkappa h) \\ &- i\rho (\omega + \varkappa v) (A \operatorname{sh} \varkappa h + B \operatorname{ch} \varkappa h) - \varkappa^2 T_1 b_1 + \rho_1 (\omega + \\ &\quad \varkappa v_1)^2 b_1 = 0 \\ &i\rho (\omega + \varkappa v) (-A \operatorname{sh} \varkappa h + B \operatorname{ch} \varkappa h) - \varkappa^2 T_2 b_2 + \rho_2 (\omega + \varkappa v_2)^2 b_2 = 0 \end{aligned} \tag{1.10}$$

The condition for a non-zero solution of the system to exist, the dispersion equation, has the form

$$\begin{aligned} &[\rho (\omega + \varkappa v)^2 \operatorname{th} \varkappa h + \varkappa \rho_1 (\omega + \varkappa v_1)^2 - T_1 \varkappa^3] [\rho (\omega + \varkappa v)^2 + \\ &\quad \varkappa \rho_2 (\omega + \varkappa v_2)^2 \operatorname{th} \varkappa h - T_2 \varkappa^3 \operatorname{th} \varkappa h] + [\rho (\omega + \varkappa v)^2 \operatorname{th} xh + \\ &\quad \varkappa \rho_2 (\omega + \varkappa v_2)^2 - T_2 \varkappa^3] [\rho (\omega + \varkappa v)^2 + \varkappa \rho_1 (\omega + \varkappa v_1)^2 \operatorname{th} \varkappa h - \\ &\quad T_1 \varkappa^3 \operatorname{th} \varkappa h] = 0 \end{aligned} \tag{1.11}$$

For small $\varkappa$ Eq.(1.11) can be simplified. Since the phase velocity of the wave $c = \omega/\varkappa$ is a finite quantity, it is convenient to ascribe a first order of smallness in ω to $\varkappa$. Using the relationship $\tanh \varkappa h \approx \varkappa h$ and also taking into account that the second and third components in the second and fourth square brackets have a higher order of smallness than the first, we obtain approximately

$$\begin{aligned} &\varkappa h \rho (\omega + \varkappa v)^2 [2\varkappa \rho h (\omega + \varkappa v)^2 + \varkappa \rho_1 (\omega + \varkappa v_1)^2 + \\ &\quad \varkappa \rho_2 (\omega + \varkappa v_2)^2 - T_1 \varkappa^3 - T_2 \varkappa^3] = 0 \end{aligned}$$

Equating the expression in square brackets to zero, we obtain

$$\begin{aligned} &(2\rho h + \rho_1 + \rho_2) \omega^2 + 2 (2\rho h v + \rho_1 v_1 + \rho_1 v_1 + \rho_2 v_2) \varkappa \omega + \\ &\quad (2\rho h v^2 + \rho_1 v_1^2 + \rho_2 v_2^2 - T_1 - T_2) \varkappa^2 = 0 \end{aligned} \tag{1.12}$$

which is equivalent to (1.6) for a given specific system when taking account of (1.9).

Since the wave velocity is independent of its length in this approximation, the system can be described by a one-dimensional second-order equation. It is seen that a system of n membranes separated by $n - 1$ fluid layers possesses the same properties. Longitudinal motions of parts of such a system merely result in different "forward" and "backward" wave propagation velocities, to a first approximation.

It is convenient to study the form of the next approximation in a rather simpler example: one membrane and fluid layer $(\rho_2 = T_2 = 0)$. Then we have instead of (1.11)

$$\rho (\omega + \varkappa v)^2 \operatorname{th} 2\varkappa h + \varkappa \rho_1 (\omega + \varkappa v_1)^2 - T_1 \varkappa^3 = 0$$

Expanding $\tanh 2\varkappa h$ in series, we obtain the next approximation after (1.12) including fourth-order terms. A one-dimensional differential equation of the same order describing the dispersion effects corresponds to it. Higher approximations can also be considered.

A characteristic property of vibrational systems described by (1.6) is their asymmetry, as expressed in the different wave velocities in the forward and backward directions. If the solution of (1.6) is taken in the form

$$u = f_1 (x - c_1 t) + f_2 (x + c_2 t) \tag{1.13}$$

we obtain expressions c_1 and c_2

$$c_{1,2} = (\pm P + \sqrt{P^2 - R (K - \theta)})/R$$

It is assumed that $P^2 - R (K - \theta) > 0$, i.e., the hyperbolicity of (1.6). For moving strings $c_{1,2} = c_0 \pm v$, where (1.3) remains hyperbolic for all v.

For the hose model (1.5) (we set $\Pi = 0$) it is customary to speak about two critical flud flow velocities. For the first critical velocity $v_* = \sqrt{T_1/\rho_1}$ the coefficient of u_{xx} vanishes and an infinite set of equilibrium modes $u = \varphi (x)$ appears, where φ is an arbitrary function. For $v > v_*$ we sepak about post-critical motions. The second critical value $v_{**} = (1 + \rho_1/\rho_2) \sqrt{T_1/\rho_1}$ corresponds to the passage from a hyperbolic to an elliptic equation. The terminology can be retained for the general case. It can always be assumed that $c_1 > 0$, $c_1 > c_2$, where $c_2 = 0$ corresponds to the first critical value. In the post-critical case, the waves move in just one direction for $c_2 < 0$. The equation $c_2 = -c_1$ corresponds to the second critical value.

The wave equation having the general solution (1.13) can be written in the form

$$u_{tt} + (c_1 - c_2)\, u_{tx} - c_1 c_2 u_{xx} = 0 \tag{1.14}$$

We will consider mainly the first boundary value problem for it, with the following conditions on fixed boundaries:

$$u\,(t,\,0) = u\,(t,\,l) = 0 \tag{1.15}$$

Keeping the problem of forced vibrations in mind, we also introduce the inhomogeneous equation

$$u_{tt} + (c_1 - c_2)\, u_{tx} - c_1 c_2 u_{xx} = f\,(t,\,x) \tag{1.16}$$

2. Free vibrations. To solve the boundary value problems, it is convenient to transform (1.16) into a wave equation of the form (1.1) by keeping the segment $[0,\,l]$ fixed. This can be done by selecting the new variables for $c_2 \neq 0$ in the form

$$\xi = x,\ \tau = t + \gamma x,\ \gamma = (c_1 - c_2)/2c_1 c_2 \tag{2.1}$$

Equation (1.16) becomes

$$u_{\tau\tau} - a^2 u_{\xi\xi} = f_1, \qquad f_1 = \frac{4c_1c_2}{(c_1+c_2)^2}\, f\,(\tau - \gamma\xi,\, \xi) \tag{2.2}$$

In this case the wave velocities will equal $\pm a$, where $a = 2c_1c_2/(c_1 + c_2)$. For moving strings /4/ $\gamma = v/(c_0^2 - v^2)$ and $a = c_0\,(1 - v^2/c_0^2)$.

A solution of the Cauchy problem can be constructed for (1.14) under the conditions

$$u\,(0,\,x) = \varphi\,(x),\ u_t\,(0,\,x) = \psi\,(x) \tag{2.3}$$

by appling the D'Alembert method directly to this equation

$$u\,(t,\,x) = \frac{1}{c_1 + c_2}\left[c_1\varphi\,(x + c_2 t) + c_2\varphi\,(x - c_1 t) + \int\limits_{x - c_1 t}^{x + c_2 t} \psi\,(y)\,dy\right]$$

Considering the free vibrations problem in a finite interval with conditions (1.15), it is convenient to start from the solution of the corresponding problem for (2.2)

$$u\,(\tau,\,\xi) = \operatorname{Re}\sum_{k=1}^{\infty} D_k \sin \varkappa_k \xi e^{i\omega_k\tau}, \qquad \varkappa_k = \frac{k\pi}{l}, \qquad \omega_k = a\varkappa_k \tag{2.4}$$

where Re is the symbol for the real part, $D_k = D_k' - iD_k''$ are arbitrary constants, and ω_k are the eigenfrequencies. Applying the transformation (2.1) to (2.4) we obtain

$$u\,(t,\,x) = \sum_{k=1}^{\infty} [D_k' \cos \omega_k\,(t + \gamma x) + D_k'' \sin \omega_k\,(t + \gamma x)] \sin \varkappa_k x \tag{2.5}$$

$$\omega_k = \frac{2c_1c_2}{c_1 + c_2}\,\varkappa_k$$

As we approach the first critical value $\omega_k \to 0$ and as we approach the second $\omega_k \to -\infty$. For a moving string $\omega_k = (1 - v^2/c_0^2)c_0\varkappa_k$.

Underlying the similarly between moving body vibrations and fixed string vibrations noted in the literature namely the reality of the eigenfrequencies; where the eigenfrequencies are "dynamic modes" representable as the sum of two travelling waves and passing to the limit $v \to 0\ (c_2 \to c_1)$ in the natural vibrations of a fixed string, are the properties of the transformation (2.1) that transfers the elementary solutions of the wave equation $\sin \varkappa_k \xi e^{i\omega_k\tau}$ into the solution of (1.3) and (1.5) keeping the values of ω_k unchanged.

If the initial conditions are given for $t = -\gamma x$, the coefficients D_k will be determined from the system

$$\varphi_1\,(x) = u\,(-\gamma x,\,x) = \sum_{k=1}^{\infty} D_k' \sin \varkappa_k x$$

$$\psi_1\,(x) = u_t\,(-\gamma x,\,x) = \sum_{k=1}^{\infty} \omega_k D_k'' \sin \varkappa_k x$$

from the usual formulas

$$D_k' = \frac{2}{l}\int\limits_0^l \varphi_1\,(y) \sin \varkappa_k y\,dy, \qquad D_k'' = \frac{2}{\omega_k l}\int\limits_0^l \psi_1\,(y) \sin \varkappa_k y\,dy$$

If the conditions are given for $t = 0$, then D_k must be determined by using expansions in a non-orthogonal system of functions

$$\varphi(x) = u(0, x) = \sum_{k=1}^{\infty} [D_k' \cos\omega_k\gamma x + D_k'' \sin\omega_k\gamma x] \sin\varkappa_k x$$

$$\psi(x) = u_t(0, x) = \sum_{k=1}^{\infty} \omega_k [-D_k' \sin\omega_k\gamma x + D_k'' \cos\omega_k\gamma x] \sin\varkappa_k x$$

We examine the question of the energy of free vibrations. For $F = 0$ and conditions (1.5), the equation for the vibrations of a moving string (1.2) can be obtained by varying the functional $J = \int L dt$ with the Lagrange functions

$$L = \frac{1}{2} \int_0^l [\rho(u_t + vu_x)^2 - Tu_x^2]\, dx$$

It is natural to determine the vibration energy from the formula $E = u_t \partial L / \partial u_t - L$, which yields

$$E = \frac{1}{2} \rho \int_0^l [u_t^2 + (c_0^2 - v^2) u_x^2]\, dx$$

Similarly for (1.14) in a calcilation per unit density

$$L = \frac{1}{2} \int_0^l [u_t^2 + (c_1 - c_2) u_t u_x - c_1 c_2 u_x^2]\, dx$$

$$E = \frac{1}{2} \int_0^l (u_t^2 + c_1 c_2 u_x^2)\, dx$$

The energy is positive-definite for the subcritical velocities. It is easy to verify that the energy density $(u_t^2 + c_1c_2u_x^2)/2$ and its flux, given by the expression $(c_1 - c_2) u_t^2/2 - c_1c_2u_tu_x$, satisfy the equation of continuity. Under conditions (1.15) the energy flux on the section boundaries equals zero, the system is closed, and its energy is conserved.

3. Forced Vibrations. Steady-state mode problems play a major role in the analysis of forced vibrations. For a fixed string it is obtained as the limit as $t \to \infty$ for the solution of an equation with the dissipative term νu_t, $\nu > 0$

$$u_{tt} + \nu u_t - c_0^2 u_{xx} = f \tag{3.1}$$

where ν tends to zero in the solution obtained. We obtain the vibrations equation for a moving string from (3.1) by the same replacement of the time derivative with respect to time by the substantive derivative, which is used to obtain (1.2) from (1.1)

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2 u + \nu\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) u - c_0^2 \frac{\partial^2 u}{\partial x^2} = f \tag{3.2}$$

The dissipative term is also introduced in /5/ as the product of a positive constant and the substantive derivative, where vibrations of a chain transmission are investigated due to the effect of perturbations moving from the boundary. The solution of (3.2) obtained in /5/ yields a decrease in the amplitude of the resonance vibrations as the chain velocity increases, which agrees with experiment while theory predicts an increase in the amplitude for the equation with the term νu_t. Transforming (3.2) to the variables τ, ξ we obtain

$$u_{\tau\tau} + \nu [u_\tau + v(1 - v^2/c_0^2) u_\xi] - a^2 u_{\xi\xi} = (1 - v^2/c_0^2) f$$

The dissipative term for (1.16) can be taken in the form $\nu(u_t + (c_1 - c_2) u_x/2)$ and the transformed equation will be

$$u_{\tau\tau} + \nu \left[u_\tau + a\frac{c_1 - c_2}{c_1 + c_2} u_\xi\right] - a^2 u_{\xi\xi} = f_1$$

When $f = f_1 = 0$ the initial perturbations on the final segment damp out for the mentioned selection of the dissipative term for all, including the post-critical, modes of motion.

Consider the problem of the steady vibrations of a moving body subjected to a force varying sinusoidally with time by taking the same problem for a fixed string /6/ as the initial problem

$$u_{\tau\tau} - a^2 u_{\xi\xi} = f_0(\xi) e^{i\omega\tau},\quad u(\tau, 0) = u(\tau, l) = 0 \tag{3.3}$$

In the non-resonance case the solution has the form $u = U(\xi) e^{i\omega\tau}$, where

$$U = \frac{\sin\varkappa\xi}{\varkappa \sin\varkappa l} \int_0^l f_0(y) \sin\varkappa(\xi - y)\, dy - \frac{1}{\varkappa} \int_0^\xi f_0(y) \sin\varkappa(\xi - y)\, dy$$

If the force $f = f_*(x) e^{i\omega\tau}$ is taken in (1.16), then after going over to the variables τ, ξ we will have

$$f_0(\xi) = 4c_1c_2(c_1 + c_2)^{-2} f_*(\xi) e^{-i\omega\gamma\xi} \tag{3.4}$$

Taking (3.4) into account, we will write the solution of the problem in the form

$$u = \frac{4c_1c_2}{(c_1+c_2)^2}\left\{\frac{\sin\varkappa x}{\varkappa\sin\varkappa l}\int_0^l f_*(y) e^{-i\omega\gamma y}\sin\varkappa(x-y)\,dy - \right.$$
$$\left.\frac{1}{\varkappa}\int_0^x f_*(y) e^{-i\omega\gamma y}\sin\varkappa(x-y)\,dy\right\} e^{i\omega(t+\gamma x)}$$

It can be concluded from this expression that near the first critical value of the velocity the amplitude of forced vibrations will decrease; $u \to 0$ as $c_2 \to 0$. However, $\varkappa \to \infty$ here and there is some doubt about the applicability of the one-dimensional approach (the influence of small stiffness on the string vibration for $v = c_0$ is studied in /7/). The amplitude tends to infinity as one approaches the second critical value $(c_2 \to -c_1)$.

We will now examine the resonance case $\omega = \omega_n = n\pi a/l$. If $f_0(\xi)$ is orthogonal to the n-th eigenfunction, i.e.,

$$\int_0^l f_0(y)\sin\varkappa_n y\,dy = 0$$

then the solution of problem (3.3) will be

$$u = -\frac{e^{i\omega_n\tau}}{\varkappa_n}\int_0^{\xi} f_0(y)\sin\varkappa_n(\xi - y)\,dy$$

The solution of problem (15, 16) is obtained after substituting (2.1) and (3.4) into this formula. It follows from the orthogonality condition that in this case $f_*(x)$ is representable in the form

$$f_*(x) = e^{i\omega_n\gamma x}\sum_{k\neq n} C_k \sin\varkappa_k x$$

C_k are complex constants. In particular, the expression $f_* = f\delta(x - x_0)$ holds for a point force, where x_0 is the zero of the function, and $\sin n\pi x/l$ is the same result as for a fixed string.

In the absence of orthogonality the solution of the problem (we present only the growing term) is obtained from the solution of the equation

$$u_{\tau\tau} - a^2 u_{\xi\xi} = A_n \sin\varkappa_n\xi e^{i\omega_n\tau}, \quad A_n = \frac{2}{l}\int_0^l f_0(y)\sin\varkappa_n y\,dy$$

and has the form

$$u(t, x) = -\frac{iA_n}{\omega_n}(t + \gamma x)\sin\varkappa_n x e^{i\omega_n(t+\gamma x)}$$

In conclusion, we note that the main results of this work can be extended formally into the post-critical case. At the same time it is difficult to give a physical interpretation of the solution of boundary value problems in the absence of waves reflected from the boundaries. It is also not clear to what degree one-dimensional models can generally be used to describe post-critical vibrations. In this connection, the inadequate experimental study of the vibrations of bodies moving at velocities comparable to and even more so, exceeding the velocity of wave propagation should be mentioned.

REFERENCES

1. GOROSHKO O.A. and SAVIN G.N., Introduction to the Mechanics of Deformable One-Dimensional Bodies of Variable Length. Naukova Dumka, Kiev, 1971.
2. SVETLITSKII V.A., Mechanics of Pipelines and Hoses. Mashinostroenie. Moscow, 1982.
3. LAMB H., Hydrodynamics /Russian translation/, Gostekhizdat, Moscow-Leningrad, 1947.
4. SACK R.A. Transverse oscillations in travelling strings, Brit. J. Appl. Phys., Vol.5, No.6, 1954.
5. MAHALINGAM S., Transverse vibrations of power transmission chains, Brit. J. Appl. Phys., Vol.8, No.4, 1957.
6. BUDAK B.M., SAMARSKII A.A. and TIKHONOV A.N., Collection of Problems on Mathematical Physics. Nauka, Moscow, 1972.

7. KOZHESHNIK YA., Transverse vibrations of stressed flexible transmission linkages. Theory of Machines and Mechanisms. Nauka, Moscow, 1976.

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THE DYNAMICS OF THE COLLISION BETWEEN A RIGID BODY AND A FLEXIBLE STRING AND MEMBRANE*

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An exact analysis of the boundary conditions at the point where an element of an absolutely flexible string or membrane meets the surface of a rigid body colliding with it as the "supersonic" velocity of the rigid body, the formulation of the problem concerning such a collision, accompanied by the tearing of the string of the rupture of the membrane, and the construction of its solution for the selfsimilar impact mode with constant velocity are given.

The principles of the mathematical theory for the collision of a solid with flexible structures in the form of strings or membranes were laid by Rakhmatullin /1/. A number of interesting results were obtained, but certain questions of the theory have not been clarified with finality. In particular, no final deductions were made regarding the set of possible formulations of the boundary conditions at the point where the element of flexible construction meets the surface of the solid. There was also no formulation of the problem of a collision accompanied by rupture of the flexible structure. The solution of these two questions is given below.

1. We will limit ourselves to examining the case when the material of the flexible structure is described by a linear law of elasticity in terms of conditional stresses while the collisions are such that the point of encounter of the structure element and the body surface is displaced at "supersonic" velocity over the structure, i.e., at a velocity exceeding the velocity of elastic wave propagation. Since abrupt bending of the structure (Fig.1) occurs at the point of encounter, i.e., a "jump" change in the momentum vector of the structure element as well as of its state of stress and strain is observed, a local reaction of the impacting body surface will be developed at this point which is modeled by a concentrated force. Taking the above into account for an idealized consideration of the problem, when the flexible structure (string or membrane) is considered as a one- or two-dimensional deformable continuum, the mechanics of the events in a small neighbourhood of the "break" point of the structure is modelled by introducing a "wave of strong discontinuity", i.e., a scheme with a jump-like change in the mechanical parameters at this point is introduced.

The Lagrange and Euler coordinates r and u, measured, respectively, along the strucutre from the point of its first contact with the impacting body and along the surface of this body, are introduced as the mechanical characteristics of the process in the one-dimensional case, as are the radial and azimuthal stresses $\bar{\sigma}_r, \bar{\sigma}_\theta$ (in the case of a string, one stress is along the filament $\bar{\sigma}_r = \bar{\sigma}_x$), corresponding to the strains

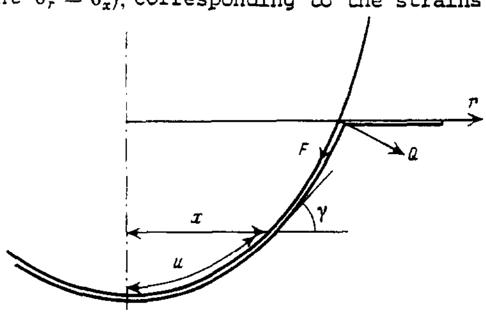

Fig.1

*Prikl.Matem.Mekhan., 49, 1, 85-93, 1985